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THE TRANSIENT ANALYSIS OF A SUBSURFACE INCLINED CRACK SUBJECTED TO A BURIED DYNAMIC DILATATIONAL SOURCE

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Abstract—The transient response of a subsurface inclined crack subjected to a buried dilatational source is analysed in this study. This investigation will be helpful in understanding the problems of seismology and waves generated by thermal loading in a cracked body. The solutions are determined by superposition of the fundamental solution in the Laplace transform domain. The required fundamental solution is the exponentially distributed traction on crack faces proposed by Tsai and Ma [1992a, *ASME J. Appl. Mech.* (in press)]. The exact closed form solutions of the time dependent dynamic stress intensity factors are obtained. These solutions are valid in the time interval from initial loading until the first wave scattered by the crack tip returns to the crack tip after reflection from the free surface. The direction of the crack propagation is predicted by different fracture criteria from the available stress intensity factors.

1. INTRODUCTION

The phenomenon of an earthquake can be considered as a complicated elastic wave propagation process. Some elastic waves travel through the interior of the earth, and some over its surface, while others are reflected from its surface. These waves give us the most detailed information about the internal structure of the earth. If there is a pre-existing fault, it will disturb the propagation wave and make the theoretical analysis much more difficult than in a homogeneous medium.

The mathematical analysis of dynamic fracture is extremely difficult. The existing solutions have limited applicability, since they rarely incorporate the effects of external boundaries. Garvin (1956) analysed displacemens on a plane surface subjected to a buried dilatational source. His results were limited to the homogeneous half-space problem. Harris (1980) solved the interaction of a dilatational wave with the semi-infinite crack in an unbounded medium. These results were also constrained to the unbounded medium. In this paper, a two-dimensional analysis of a subsurface inclined crack subjected to a buried dilatational source is studied. This may shed some light on the investigation of seismology. In analysing this problem, the reflections and diffractions of stress waves by the material boundaries and by the crack itself must be taken into account. The dilatational source can be generated by an explosive impulse and a thermal expansion effect. In many applications, the characteristic time for heat input is much larger than the characteristic mechanical response time, so that the inertia effect is small. If the inertia effect is significant, then the stress wave generated by heating must be considered. The dynamic dilatational source can be used to simulate a thermal stress wave generated by heating. If the distance from the crack tip to the free surface is large, the solution obtained in this study can be reduced to one for a semi-infinite crack in an unbounded medium.

In a conventional analysis of a semi-infinite crack in an unbounded medium subjected to dynamic loading, the complete solution can be obtained by integral transform methods together with direct application of the Wiener–Hopf technique (Noble, 1958) and the Cagniard–de Hoop method (de Hoop, 1958) of Laplace inversion. The transient analysis of an elastic solid containing a half-plane crack subjected to concentrated impact loading of the crack faces was dislocation climbing along the crack tip line with a constant speed to construct the solution for the transient response. Based on this methodology, Brock's (1982, 1984) and Brock *et al.*'s (1985) analyses investigated a series of problems of a semi-infinite crack subjected to impact loading. Recently, Lee and Freund (1990) analysed the fracture initiation of an edge-cracked plate subjected to an asymmetric impact by a similar procedure. Ma and Hou (1990a, b) analysed the characteristic time required for the transient response of a semi-infinte crack subjected to dynamic loading to approach the corresponding static solution. We consider the problem in this study to be the transient response of an elastic half-plane with a semi-infinite inclined crack subjected to a buried dynamic dilatational loading. The methods proposed by Freund (1974) and Brock *et al.* (1985) are very difficult to apply and do not provide an easy solution to the problem investigated in this study. An alternative fundamental solution proposed by Tsai and Ma (1992a) is used to overcome this difficulty. Finally, the possibility of crack surface interpenetration is evaluated by the analysis of crack opening displacement.

2. FUNDAMENTAL SOLUTIONS

As usual in problems of the type considered here, superposition of solutions plays a significant role. The solutions of the problems considered in this study can be determined by superposition of the following problems A and B. Problem A treats the dynamic force acting on the same semi-infinite half-plane without a crack while inducing a traction on the planes which will eventually define the initial crack faces. In problem B, an infinite body containing a semi-infinite crack is considered in which the faces are subjected to tractions which are equal and opposite to those on the corresponding planes in problem A. The sum of the solutions to problems A and B is the solution to the problem of diffraction of incident waves by a stationary inclined crack.

From the physical viewpoint, reflected and diffracted fields are generated to eliminate the stress induced by incident waves on the traction-free boundary. For most of the dynamic problems, the incident waves can be represented in the exponentially functional form in the Laplace transform domain of time. Unlike the usual superposition method which is performed in the time domain, the superposition scheme proposed in this study is performed in the Laplace transform domain.

Consider plane strain deformation of a semi-infinite crack contained in an unbounded medium. An exponentially distributed traction in the Laplace transform domain is applied to the crack faces. The traction force can be divided into a normal force (mode I) and a tangential force (mode II). Because of symmetry with respect to the plane $x_2 = 0$, the problem can be viewed as a half-plane problem with the material occupying the region $x_2 \geqslant 0$, subjected to the boundary conditions:

mode I

$$\begin{aligned} \bar{\sigma}_{22}(x_1,0,p) &= \mathrm{e}^{p\eta x_1}, \quad -\infty < x_1 \leqslant 0, \\ \bar{\sigma}_{12}(x_1,0,p) &= 0, \quad -\infty < x_1 < \infty, \\ \bar{u}_2(x_1,0,p) &= 0, \quad 0 \leqslant x_1 < \infty, \end{aligned} \tag{1}$$

and mode II

$$\begin{aligned} \bar{\sigma}_{22}(x_1,0,p) &= 0, \quad -\infty < x_1 < \infty, \\ \bar{\sigma}_{12}(x_1,0,p) &= \mathrm{e}^{p\eta x_1}, \quad -\infty < x_1 \leqslant 0, \\ \bar{u}_1(x_1,0,p) &= 0, \quad 0 \leqslant x_1 < \infty, \end{aligned} \tag{2}$$

where p is the Laplace transform parameter and η is a constant. The overbar symbol is used to denote the transform on time t. The final results of the dynamic mode I, II stress intensity factors and mode I crack surface opening displacement in the Laplace transform domain are

$$\bar{K}_{\rm I} = -\sqrt{\frac{2}{p}}\,K_{\rm I}^{\rm F}(\eta), \quad \bar{K}_{\rm II} = -\sqrt{\frac{2}{p}}\,K_{\rm II}^{\rm F}(\eta),$$

$$\Delta\bar{u}_2 = \frac{1}{2\pi {\rm i}\mu p}\int_{\Gamma^\lambda} \Delta u_2^{\rm F}(\lambda,\eta)\,{\rm e}^{p\lambda x_1}\,{\rm d}\lambda, \tag{3}$$

where

$$K_{\rm I}^{\rm F}(\eta) = \frac{(a+\eta)^{1/2}}{(c+\eta)S_+(\eta)}, \quad K_{\rm II}^{\rm F}(\eta) = \frac{(b+\eta)^{1/2}}{(c+\eta)S_+(\eta)},$$

$$\Delta u_2^{\rm F}(\lambda,\eta) = \frac{-b^2(a-\lambda)^{1/2}(a+\eta)^{1/2}}{(b^2-a^2)(\eta-\lambda)(c-\lambda)(c+\eta)S_-(\lambda)S_+(\lambda)},$$

$$S_\pm(\eta) = \exp\left(\frac{-1}{\pi}\int_a^b \tan^{-1}\left[\frac{4\lambda^2(\lambda^2-a^2)^{1/2}(b^2-\lambda^2)^{1/2}}{(b^2-2\lambda^2)^2}\right]\frac{{\rm d}\lambda}{\lambda\pm\eta}\right),$$

and $a = \sqrt{\rho/(\gamma+2\mu)}$, $b = \sqrt{\rho/\mu}$, a, b and c are the slownesses of the longitudinal wave, shear wave and Rayleigh wave, μ and ρ are the shear modulus and mass density, and γ is the Lamé elastic constant. Γ^λ is the usual inversion path for the two-sided Laplace transform from $\lambda_1-{\rm i}\infty$ to $\lambda_1+{\rm i}\infty$, and λ_1 is a real number and is located in the interval $|\lambda_1| < a$.

3. SUBSURFACE CRACK SUBJECTED TO BURIED DILATATIONAL SOURCE

The problems of seismology and thermal stress wave interaction with a crack can be idealized by a two-dimensional subsurface crack subjected to a dynamic dilatational source. The geometric configuration considered in this paper is an inclined semi-infinite crack located under the surface of a half-plane as shown in Fig. 1. The incident wave generated by the dynamic dilatational source will be diffracted from the crack tip and reflected from the free surface. In this study, we focus our attention mainly on the dynamic stress intensity factor. The results of the dynamic stress intensity factor obtained in this study are valid in the time before the diffracted wave generated from the crack returns to the crack tip after reflection from the free surface. The origins of the two coordinate systems $(\bar{x}_1, \bar{x}_2)$ and (x_1, x_2) are located at the plane surface and the crack tip, respectively. The planar crack lies in the plane $x_2 = 0$, $x_1 < 0$ and the inclined angle of the crack is θ. The coordinate transforms and stress relations between these two systems are:

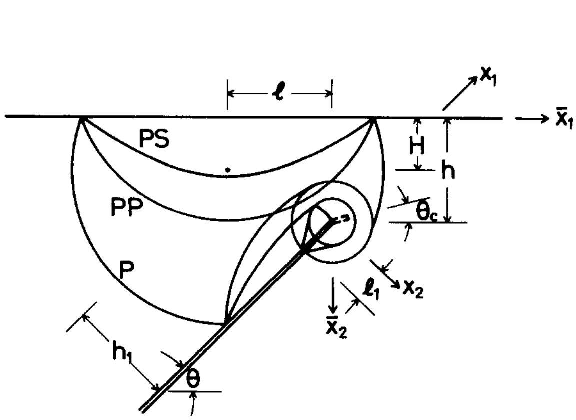

Fig. 1. Configuration, coordinate system and wave fronts of a sub-surface crack subjected to a dynamic dilatational source.

$$\bar{x}_1 = x_1 \cos\theta + x_2 \sin\theta,$$
$$\bar{x}_2 = -x_1 \sin\theta + x_2 \cos\theta + h, \tag{4}$$

$$\sigma_{22} = \sigma_{\bar{1}\bar{1}} \sin^2\theta + \sigma_{\bar{2}\bar{2}} \cos^2\theta + \sigma_{\bar{1}\bar{2}} \sin 2\theta,$$
$$\sigma_{12} = \tfrac{1}{2}(\sigma_{\bar{1}\bar{1}} - \sigma_{\bar{2}\bar{2}}) \sin 2\theta + \sigma_{\bar{1}\bar{2}} \cos 2\theta, \tag{5}$$

where h is the vertical distance from the crack tip to the plane surface.

A dilatational source applied suddenly in an infinite medium can be represented by the wave equation of longitudinal potential ϕ as follows

$$\nabla^2\phi - a^2\ddot{\phi} = \frac{\delta(r)}{r} H(t). \tag{6}$$

The corresponding longitudinal potential ϕ_0 can be expressed in the Laplace transform domain as follows

$$\bar{\phi}_0(r,\theta,p) = -\frac{K_0(par)}{p}, \tag{7}$$

where K_0 is the modified Bessel function of the second kind of order zero. Now, consider a buried dilatational source located at $\bar{x}_1 = l$, $\bar{x}_2 = H$ (or $x_1 = l_1$, $x_2 = -h_1$) applied suddenly at time $t = 0$. The stresses in the region $x_2 + h_1 > 0$ induced by this dilatational source in the Laplace transform domain are

$$\bar{\sigma}_{22}(x_1,x_2,p) = \frac{\mu p}{2\pi i}\int_{\Gamma^\lambda} A_1(\lambda)\, e^{-p\alpha(x_2+h_1)\delta + p\lambda(x_1-l_1)}\, d\lambda, \tag{8}$$

$$\bar{\sigma}_{12}(x_1,x_2,p) = \frac{\mu p}{2\pi i}\int_{\Gamma^\lambda} A_2(\lambda)\, e^{-p\alpha(x_2+h_1) + p\lambda(x_1-l_1)}\, d\lambda, \tag{9}$$

where

$$A_1(\lambda) = -\frac{\pi(b^2-2\lambda^2)}{\alpha}, \quad A_2(\lambda) = 2\pi\lambda, \quad \alpha = (a^2-\lambda^2)^{1/2}.$$

The incident longitudinal P wave will be reflected and diffracted from the crack. There will also be PP and PS waves reflected from the free surface, which will arrive at the crack tip at a later time. These two situations will be investigated separately in the following analysis. The interaction of the dilatational P wave with the semi-infinite crack in an unbounded medium is considered first. The traction applied on the crack faces to eliminate the incident P wave can be obtained from (8) and (9) by letting $x_2 = 0$:

$$\bar{\sigma}_{22} = -\frac{\mu p}{2\pi i}\int_{\Gamma^\lambda} A_1(\lambda)\, e^{-p\alpha h_1 - p\lambda l_1 + p\lambda x_1}\, d\lambda, \tag{10}$$

$$\bar{\sigma}_{12} = -\frac{\mu p}{2\pi i}\int_{\Gamma^\lambda} A_2(\lambda)\, e^{-p\alpha h_1 - p\lambda l_1 + p\lambda x_1}\, d\lambda. \tag{11}$$

It is shown in (10) and (11) that tractions on the crack surfaces are represented by the exponential function $e^{p\lambda x_1}$, while the stress intensity factors of an applied traction $e^{p\eta x_1}$ are expressed in (3). So, the induced stress intensity factors of (10) and (11) in the Laplace transform domain can be constructed by superposition of the fundamental solution obtained in Section 2. The results are:

$$\bar{K}_{\mathrm{I}}^{\mathrm{I}} = \frac{\mu\sqrt{p}}{\sqrt{2\pi i}} \int_{\Gamma^\lambda} A_1(\lambda) K_{\mathrm{I}}^{\mathrm{F}}(\lambda)\, \mathrm{e}^{-p\alpha h_1 - p\lambda l_1}\, \mathrm{d}\lambda, \tag{12}$$

$$\bar{K}_{\mathrm{II}}^{\mathrm{I}} = \frac{\mu\sqrt{p}}{\sqrt{2\pi i}} \int_{\Gamma^\lambda} A_2(\lambda) K_{\mathrm{II}}^{\mathrm{F}}(\lambda)\, \mathrm{e}^{-p\alpha h_1 - p\lambda l_1}\, \mathrm{d}\lambda. \tag{13}$$

The stress intensity factors in the time domain can be obtained by employing Cagniard's method of Laplace inversion:

$$K_{\mathrm{I}}^{\mathrm{I}}(t) = \frac{\mu\sqrt{2}}{\pi\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{ar_1}^{t} \frac{1}{\sqrt{t-\tau}} \mathrm{Im}\left[A_1(\lambda_1) K_{\mathrm{I}}^{\mathrm{F}}(\lambda_1) \frac{\partial \lambda_1}{\partial \tau} \right] \mathrm{d}\tau, \tag{14}$$

$$K_{\mathrm{II}}^{\mathrm{I}}(t) = \frac{\mu\sqrt{2}}{\pi\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{ar_1}^{t} \frac{1}{\sqrt{t-\tau}} \mathrm{Im}\left[A_2(\lambda_1) K_{\mathrm{II}}^{\mathrm{F}}(\lambda_1) \frac{\partial \lambda_1}{\partial \tau} \right] \mathrm{d}\tau, \tag{15}$$

where

$$\lambda_1 = \frac{\tau}{r_1} \cos\theta_1 + \mathrm{i}\sqrt{\frac{\tau^2}{r_1^2} - a^2} \sin\theta_1,$$

$$r_1^2 = l_1^2 + h_1^2, \quad \cos\theta_1 = \frac{l_1}{r_1}, \quad \sin\theta_1 = \frac{h_1}{r_1}. \tag{16}$$

The influence on the stress intensity factor of reflected PP and PS waves is considered now. The half-plane problem (without a crack) subjected to dynamic dilatational loading has been solved by Garvin (1956). Here we use an alternate formulation by Tsai and Ma (1991) that is more in keeping with the analytical techniques presented herein. The reflected PP and PS waves are generated at the surface of the half-plane in order to eliminate the stress induced on the free surface by the incident wave. The stresses of the incident wave in the region $H - \bar{x}_2 > 0$ can be expressed in the Laplace transform domain as follows:

$$\bar{\sigma}_{\bar{2}\bar{2}}(\bar{x}_1, \bar{x}_2, p) = \frac{\mu p}{2\pi i} \int_{\Gamma^\lambda} A_1(\lambda)\, \mathrm{e}^{-p\alpha(H-\bar{x}_2) + p\lambda(\bar{x}_1 - l)}\, \mathrm{d}\lambda, \tag{17}$$

$$\bar{\sigma}_{\bar{1}\bar{2}}(\bar{x}_1, \bar{x}_2, p) = \frac{\mu p}{2\pi i} \int_{\Gamma^\lambda} -A_2(\lambda)\, \mathrm{e}^{-p\alpha(H-\bar{x}_2) + p\lambda(\bar{x}_1 - l)}\, \mathrm{d}\lambda. \tag{18}$$

Setting $\bar{x}_2 = 0$ in (17) and (18), the traction to be cancelled in the free surface can also be represented in the exponential form. Hence, the fundamental solutions to the exponentially distributed loading on the half-plane surface obtained by Tsai and Ma (1991) can be used to construct the reflected field. Omitting details, the final results of the reflected PP and PS waves in the Laplace transform domain are presented as follows:

$$\bar{\sigma}_{22}(\bar{x}_1, \bar{x}_2, p) = \frac{\mu p}{2i} \int_{\Gamma^\lambda} [A_3(\lambda)\, \mathrm{e}^{-p\alpha(\bar{x}_2 + H) + p\lambda(\bar{x}_1 - l)} + \mathrm{A}_4(\lambda)\, \mathrm{e}^{-p\beta\bar{x}_2 - p\alpha H + p\lambda(\bar{x}_1 - l)}]\, \mathrm{d}\lambda, \tag{19}$$

$$\bar{\sigma}_{12}(\bar{x}_1, \bar{x}_2, p) = \frac{\mu p}{2i} \int_{\Gamma^\lambda} [A_5(\lambda)\, \mathrm{e}^{-p\alpha(\bar{x}_2 + H) + p\lambda(\bar{x}_1 - l)} + \mathrm{A}_6(\lambda)\, \mathrm{e}^{-p\beta\bar{x}_2 - p\alpha H + p\lambda(\bar{x}_1 - l)}]\, \mathrm{d}\lambda, \tag{20}$$

where

$$A_3(\lambda) = \frac{(b^2-2\lambda^2)^3-4\alpha\beta\lambda^2(b^2-2\lambda^2)}{R\alpha}\cos^2\theta + \frac{(b^2-2\alpha^2)((b^2-2\lambda^2)^2-4\alpha\beta\lambda^2)}{R\alpha}\sin^2\theta + \frac{-2\lambda(b^2-2\lambda^2)^2+8\alpha\beta\lambda^3}{R}\sin 2\theta,$$

$$A_4(\lambda) = \frac{8\beta\lambda^2(b^2-2\lambda^2)}{R}\cos 2\theta + \frac{4\lambda(b^2-2\lambda^2)^2}{R}\sin 2\theta,$$

$$A_5(\lambda) = \frac{1}{2}\frac{(b^2-2\alpha^2)((b^2-2\lambda^2)^2-4\alpha\beta\lambda^2)-(b^2-2\lambda^2)^3+4\alpha\beta\lambda^2(b^2-2\lambda^2)}{R\alpha}\sin 2\theta + \frac{-2\lambda(b^2-2\lambda^2)^2+8\alpha\beta\lambda^3}{R}\cos 2\theta,$$

$$A_6(\lambda) = -\frac{8\beta\lambda^2(b^2-2\lambda^2)}{R}\sin 2\theta + \frac{4\lambda(b^2-2\lambda^2)^2}{R}\cos 2\theta,$$

$$\beta = (b^2-\lambda^2)^{1/2}, \quad R = (b^2-\lambda^2)^2+4\alpha\beta\lambda^2. \tag{21}$$

The first term of both (19) and (20) represent the reflected longitudinal (PP) wave while the second term in each represent the reflected shear (PS) wave. The tractions applied on the crack suface required to satisfy the traction-free boundary condition are obtained from (19) and (20) by setting $x_2 = 0$ which yields:

$$\bar{\sigma}_{22} = -\frac{\mu p}{2i}\int_{\Gamma^\lambda}[A_3(\lambda)\,\mathrm{e}^{p(\lambda\cos\theta+\alpha\sin\theta)x_1-p\alpha(h+H)-p\lambda l} + A_4(\lambda)\,\mathrm{e}^{p(\lambda\cos\theta+\beta\sin\theta)x_1-p\beta h-p\alpha H-p\lambda l}]\,\mathrm{d}\lambda, \tag{22}$$

$$\bar{\sigma}_{12} = -\frac{\mu p}{2i}\int_{\Gamma^\lambda}[A_5(\lambda)\,\mathrm{e}^{p(\lambda\cos\theta+\alpha\sin\theta)x_1-p\alpha(h+H)-p\lambda l} + A_6(\lambda)\,\mathrm{e}^{p(\lambda\cos\theta+\beta\sin\theta)x_1-p\beta h-p\alpha H-p\lambda l}]\,\mathrm{d}\lambda. \tag{23}$$

It is shown in (22) and (23) that tractions on crack surfaces are represented by the exponential functions $\mathrm{e}^{p(\lambda\cos\theta+\alpha\sin\theta)x_1}$ and $\mathrm{e}^{p(\lambda\cos\theta+\beta\sin\theta)x_1}$. The mixed mode stress intensity factors in the Laplace transform domain due to applied tractions (22) and (23) on crack faces can be constructed by superposition. If η is replaced by $\lambda\cos\theta+\alpha\sin\theta$ and $\lambda\cos\theta+\beta\sin\theta$ and (3), (22) and (23) are combined, the final results are:

$$\bar{K}_{\mathrm{I}}^{\mathrm{R}} = \frac{\mu\sqrt{2p}}{2i}\int_{\Gamma^\lambda}[A_3(\lambda)K_{\mathrm{I}}^{\mathrm{F}}(\lambda\cos\theta+\alpha\sin\theta)\,\mathrm{e}^{-p\alpha(h+H)-p\lambda l} + A_4(\lambda)K_{\mathrm{I}}^{\mathrm{F}}(\lambda\cos\theta+\beta\sin\theta)\,\mathrm{e}^{-p\beta h-p\alpha H-p\lambda l}]\,\mathrm{d}\lambda, \tag{24}$$

$$\bar{K}_{\mathrm{II}}^{\mathrm{R}} = \frac{\mu\sqrt{2p}}{2i}\int_{\Gamma^\lambda}[A_5(\lambda)K_{\mathrm{II}}^{\mathrm{F}}(\lambda\cos\theta+\alpha\sin\theta)\,\mathrm{e}^{-p\alpha(h+H)-p\lambda l} + A_6(\lambda)K_{\mathrm{II}}^{\mathrm{F}}(\lambda\cos\theta+\beta\sin\theta)\,\mathrm{e}^{-p\beta h-p\alpha H-p\lambda l}]\,\mathrm{d}\lambda. \tag{25}$$

The stress intensity factors in the time domain can be obtained by inverting the Laplace transform using Cagniard's method. The results are:

$$\frac{\sqrt{\pi}}{\mu\sqrt{2}}K_{\mathrm{I}}^{\mathrm{R}}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{ar_2}^{t}\frac{1}{\sqrt{t-\tau}}\operatorname{Im}\left[A_3(\lambda_2)K_{\mathrm{I}}^{\mathrm{F}}(\lambda_2\cos\theta+\alpha\sin\theta)\frac{\partial\lambda_2}{\partial\tau}\right]\mathrm{d}\tau$$

$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_{T_{\mathrm{PS}}}^{t}\frac{1}{\sqrt{t-\tau}}\operatorname{Im}\left[A_4(\lambda_3)K_{\mathrm{I}}^{\mathrm{F}}(\lambda_3\cos\theta+\alpha\sin\theta)\frac{\partial\lambda_3}{\partial\tau}\right]\mathrm{d}\tau, \quad (26)$$

$$\frac{\sqrt{\pi}}{\mu\sqrt{2}}K_{\mathrm{II}}^{\mathrm{R}}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{ar_2}^{t}\frac{1}{\sqrt{t-\tau}}\operatorname{Im}\left[A_5(\lambda_2)K_{\mathrm{I}}^{\mathrm{F}}(\lambda_2\cos\theta+\alpha\sin\theta)\frac{\partial\lambda_2}{\partial\tau}\right]\mathrm{d}\tau$$

$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_{T_{\mathrm{PS}}}^{t}\frac{1}{\sqrt{t-\tau}}\operatorname{Im}\left[A_6(\lambda_3)K_{\mathrm{I}}^{\mathrm{F}}(\lambda_3\cos\theta+\alpha\sin\theta)\frac{\partial\lambda_3}{\partial\tau}\right]\mathrm{d}\tau, \quad (27)$$

where λ_3 is the root of the function $\tau=\beta h+\alpha H+\lambda l$, and

$$\lambda_2=\frac{\tau}{r_2}\cos\theta_2+\mathrm{i}\sqrt{\frac{\tau^2}{r_2^2}-a^2}\sin\theta_2,$$

$$r_2^2=l^2+(h+H)^2,\quad \cos\theta_2=\frac{l}{r_2},\quad \sin\theta_2=\frac{h+H}{r_2}. \quad (28)$$

The two terms in (26) and (27) represent the contribution of the reflected PP and PS waves, respectively. Times ar_2 and T_{PS} represent the arrival time of wave fronts for PP and PS waves at the crack tip, respectively. If the dilatational source is generated by a thermal heating variable $\Theta_0 H(t)$, stress intensity factors can be obtained by multiplying the results shown in (26) and (27) by the factor of $\bar{\alpha}\Theta_0(1+\nu)/2\pi(1-\nu)$, where $\bar{\alpha}$ is the thermal expansion coefficient and Θ_0 is a constant.

Under realistic dynamic loading conditions, it is impossible to produce a true loading with Heaviside function time dependence. Instead, the loading pulse has a finite rise time. Therefore, in order to simulate a practical explosive event, the analysis is extended to the case in which the loading pulse has a finite rise time. Suppose that at time $t=0$, the pulse is applied suddenly and the magnitude of the dilatational source increase according to the function $f(t)$ is

$$f(t)=\frac{t}{T_{\mathrm{R}}}\quad \text{for}\quad 0\leqslant t\leqslant T_{\mathrm{R}},$$

$$=1\quad \text{for}\quad T_{\mathrm{R}}<t, \quad (29)$$

where T_{R} denotes the rising time. Then, the stress intensity factors can be obtained by the superposition method

$$K'_{\mathrm{I,II}}(t)=\int_0^t\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}K_{\mathrm{I,II}}(t-\tau)\,\mathrm{d}\tau. \quad (30)$$

In this study, we also consider an explosive loading that has uniform distributed dilatational sources on a circular disk of radius r_0 with its center at $\bar{x}_1=l$, $\bar{x}_2=H$. The numerical values of the stress intensity factors for the distributed sources can be obtained using the superposition scheme. Practical structures are not only subjected to tension but also to shear loading. In mixed mode experiments, it is usually observed that crack extension takes place under an angle with respect to the original crack direction. With the mixed stress intensity factors which have been obtained, the criteria of maximum circumferential tensile stress proposed by Erdogan and Sih (1963) and minimum strain energy density proposed by Sih (1972) will be introduced to examine the crack growth direction. The maximum circumferential tensile stress criterion postulates that the crack will grow in a direction determined by the condition that the circumferential tensile stress within the asymptotic field is at a maximum, the angle between the crack line and the direction of crack growth, $\theta_2(=\theta-\theta_c)$, satisfies

$$\sin\theta_2 K_I+(3\cos\theta_2-1)K_{II}=0. \quad (31)$$

The strain energy density criterion states that crack growth takes place in the direction of minimum strain energy density. The relation for determining θ_2, the angle of crack extension, is given by:

$$(1-2v)(-2\sin\theta_2 K_I^2-4\cos\theta_2 K_I K_{II}+2\sin\theta_2 K_{II}^2)$$
$$+(\sin 2\theta_2 K_I^2+4\cos 2\theta_2 K_I K_{II}-3\sin 2\theta_2 K_{II}^2)=0. \quad (32)$$

4. CRACK SURFACE OPENING DISPLACEMENT

The crack surface opening displacement is analysed to investigate the possibility of crack surface interpenetration induced by the dynamic dilatational source. The crack opening displacement Δu_2 represents the difference in the vertical displacement for upper and lower crack surfaces. By using the superposition method, the crack opening displacement in the Laplace transform domain for the semi-infinite crack induced by the dilatational incident P wave can be expressed as follows:

$$\Delta\bar{u}_2=\frac{1}{4\pi^2}\int_{\Gamma^\eta}\int_{\Gamma^\lambda}A_1(\eta)\Delta u_2^F(\lambda,\eta)\,e^{-p\alpha(\eta)h_1-p\eta l_1+p\lambda x_1}\,d\lambda\,d\eta. \quad (33)$$

The crack opening displacement induced by reflected PP and PS waves generated from the free boundary of the half-plane can be expressed as:

$$\Delta\bar{u}_2=\frac{1}{4\pi}\int_{\Gamma^\eta}\int_{\Gamma^\lambda}[A_3(\eta)\Delta u_2^F(\lambda,\eta\cos\theta+\alpha(\eta)\sin\theta)\,e^{-p\alpha(\eta)(h+H)-p\eta l+p\lambda x_1}$$
$$+A_4(\eta)\Delta u_2^F(\lambda,\eta\cos\theta+\beta(\eta)\sin\theta)\,e^{-p\beta(\eta)h-p\alpha(\eta)H-p\eta l+p\lambda x_1}]\,d\lambda\,d\eta. \quad (34)$$

Equations (33) and (34) possess a double inversion integral which can be carried out by using the method proposed by Harris (1980). The transient effect of reflected waves induced by the crack surface and diffracted wave and Rayleigh wave generated from the crack tip can be separated from the above equations. The final form of the crack opening displacement in the time domain is very complicated and is omitted here, only the numerical results will be discussed in the next section.

5. NUMERICAL RESULTS

For the numerical calculation of the mixed dynamic stress intensity factors, Poisson's ratio v is assumed to be equal to 0.25. In this case, the ratios of the slownesses are $b=\sqrt{3}a$ and $c=1.88a$. The point explosive loadings of the Heaviside function dependence are applied at the positions $H=0.5h$, $l=-h$, 0 and h. The inclined angles of crack θ chosen for this investigation are 0°, 45° and 90°. There are nine combinations of loading position and inclined crack angle. The dynamic stress intensity factors for the time interval of interest

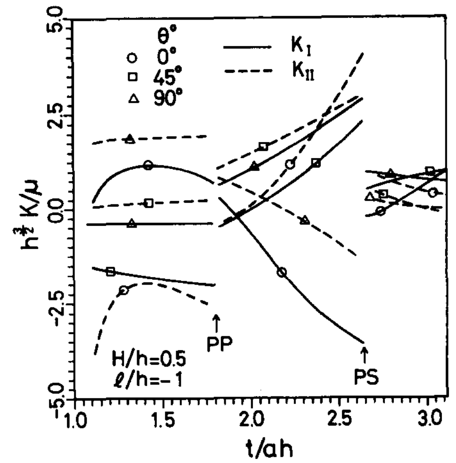

Fig. 2. Stress intensity factors K_I and K_{II} for a dilatational source at $l = -h$ and $H = 0.5h$.

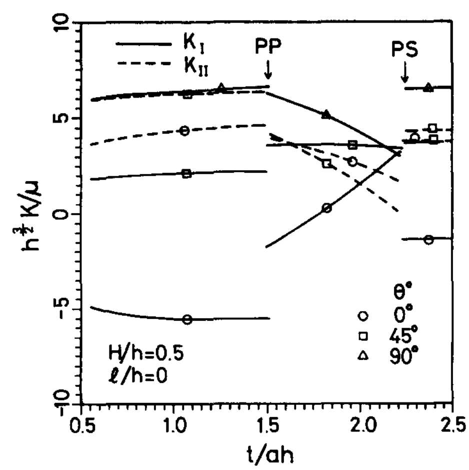

Fig. 3. Stress intensity factors K_I and K_{II} for a dilatational source at $l = 0$ and $H = 0.5h$.

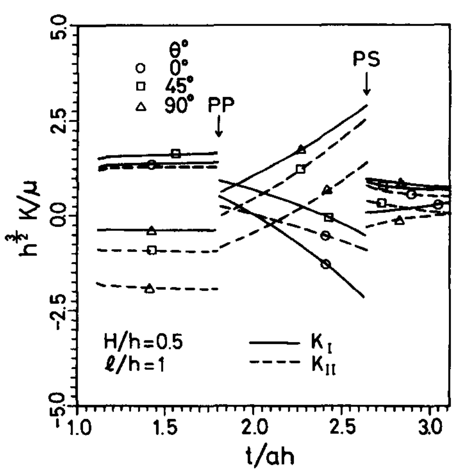

Fig. 4. Stress intensity factors K_I and K_{II} for a dilatational source at $l = h$ and $H = 0.5h$.

are shown in Figs 2–4. The crack propagation angle θ_c that satisfies (31) and (32) for the problem analysed here is shown in Figs 5 and 6.

For the case of an applied loading at the left- (right-) hand side of the crack tip, we use $l = -h$ $(l = h)$. The results obtained in this study are valid in the time period when

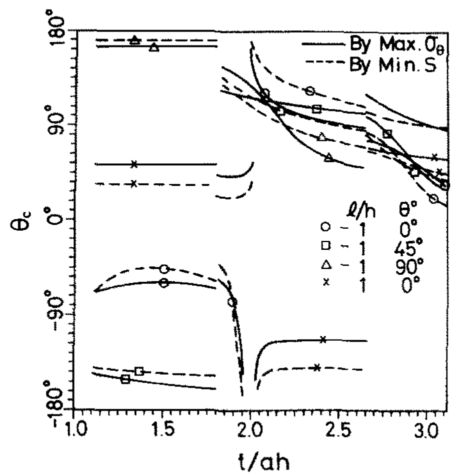

Fig. 5. Prediction of crack propagation direction for point dilatational loading.

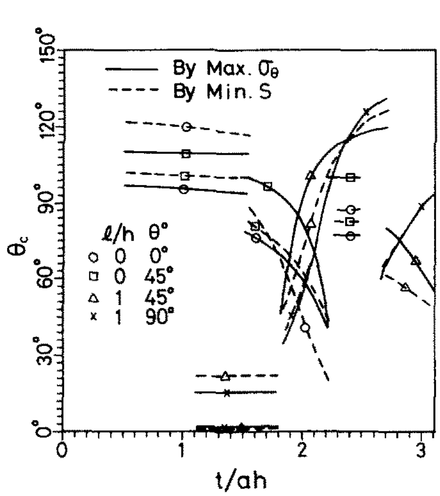

Fig. 6. Prediction of crack propagation direction for point dilatational loading.

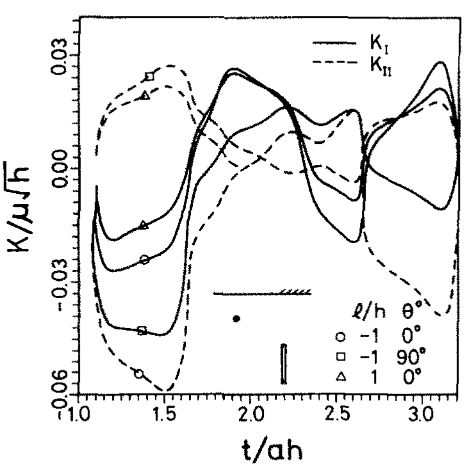

Fig. 7. Stress intensity factors K_I and K_{II} subjected to a uniformly distributed dilatational source.

diffracted waves from the crack tip reflected from the half-plane boundary have not yet returned to the crack tip, that is the time period $\sqrt{1.25} \leqslant t/ah \leqslant \sqrt{1.25}+2$. There are incident P waves generated at the explosive point and reflected PP and PS waves generated at the free surface which will strike the crack tip at different times. The normalized time

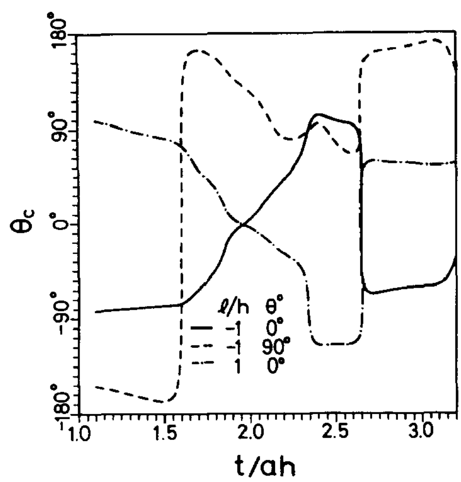

Fig. 8. Prediction of the crack propagation direction for a uniformly distributed dilatational source.

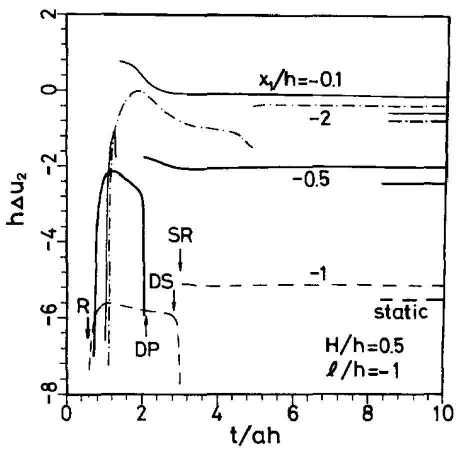

Fig. 9. Crack opening displacement Δu_2 of a semi-infinite crack in an unbounded medium induced by the dilatational source at $l = -h$ and $H = 0.5h$.

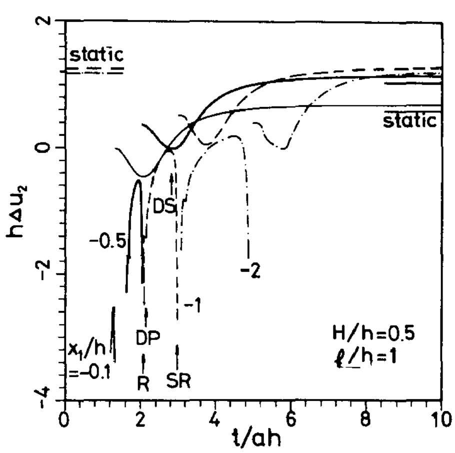

Fig. 10. Crack opening displacement Δu_2 of a semi-infinite crack in an unbounded medium induced by the dilatational source at $l = h$ and $H = 0.5h$.

t/ah for wave fronts to arrive at the crack tip are $\sqrt{1.25}$ for the P wave, $\sqrt{3.25}$ for the PP wave, and 2.642 for the PS wave. If loading is applied at $l = 0$, which is the case when the explosive source is applied directly above the crack tip. The time interval for numerical calculation in this case will be $0.5 \leqslant t/ah \leqslant 2.5$. The normalized arrival times are 0.5 for the P wave, 1.5 for the PP wave, and $0.5+\sqrt{3}$ for the PS wave. If $\theta = 90°$, the crack is

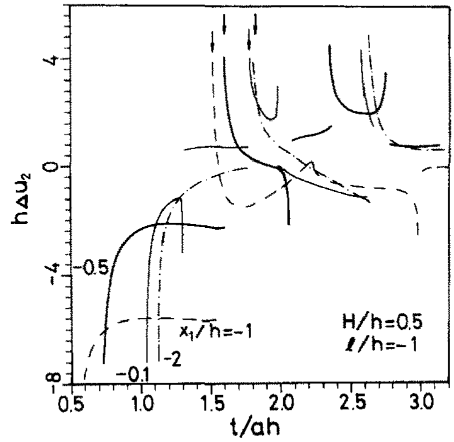

Fig. 11. Crack opening displacement Δu_2 of a subsurface crack induced by the dilatational source at $l = -h$ and $H = 0.5h$.

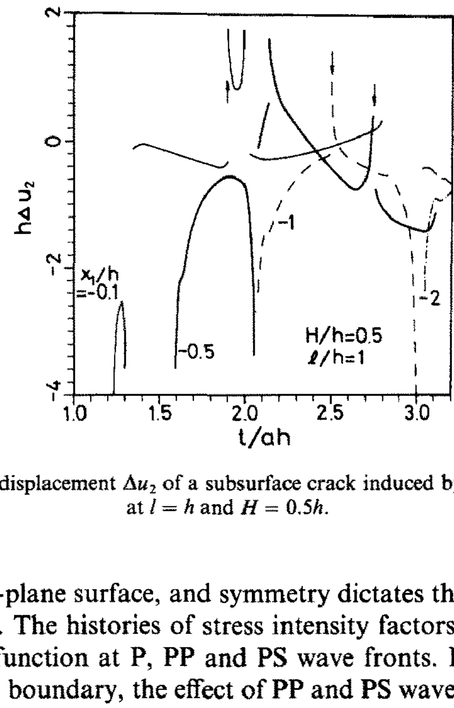

Fig. 12. Crack opening displacement Δu_2 of a subsurface crack induced by the dilatational source at $l = h$ and $H = 0.5h$.

perpendicular to the half-plane surface, and symmetry dictates that only the mode I stress intensity factor will exist. The histories of stress intensity factors show a finite jump plus an increase in the delta function at P, PP and PS wave fronts. If the buried crack is far away from the half-plane boundary, the effect of PP and PS waves can be neglected. If the stress intensity factor due to the incident P wave for time is large enough it will approach the static value obtained by using the method of Tsai and Ma (1992b).

The crack propagation angle θ_c that is predicted by maximum circumferential tensile stress and minimum strain energy density criterion for the problem analysed here is shown in Figs 5 and 6. It is concluded from these two figures that for most cases studied in this paper, the crack propagation angle θ_c is greater than zero meaning that the crack will propagate toward the half-plane surface to fracture the specimen.

In order to simulate a realistic explosive phenomenon, a rise time $T_R = 0.5ah$ is chosen for the calculation. The explosive area is assumed to be a circular disk of radius $r_0 = 0.05h$ with its center at $H = 0.5h$, $l = -h$ or h. The orientation of the crack is parallel or perpendicular to the free surface. The numerical results for the stress intensity factors are shown in Fig. 7. The result shows a continuation of the stress intensity factor at the wave fronts. The maximum circumferential tensile stress criterion is used to predict the crack propagation direction shown in Fig. 8.

In order to investigate the possibility of the crack surface interpenetration base on the transient solution obtained in this study, the crack opening displacement of a horizontal crack ($\theta = 0°$) is evaluated for a point dilatational source located at $l = -h$, h and $H = 0.5h$.

The interaction of the dilatational P wave with the semi-infinite crack in an unbounded medium is considered first. This problem can provide a clear feature of the transient effect without the influence by the half-plane boundary. For time large enough, the transient analysis of the crack opening displacement can be compared with the static solution obtained under a static loading source. The numerical results of crack opening displacement at $x_1 = -0.1h$, $-0.5h$, $-h$, $-2h$ are shown in Figs 9 and 10. The negative value for Δu_2 indicates the crack surface interpenetration and the positive value represents the crack surface opening. There are four transient waves propagating through the crack face during the full history. The incident dilatational P wave will be reflected from the crack surface as the reflected wave (R). Latterly, the incident wave will be diffracted by the crack tip and the diffracted longitudinal wave (DP) and diffracted shear wave (DS) will be generated. Finally, the Rayleigh wave (SR) arrives at the observation point. The arrival times for the above indicated waves are shown for the typical point of $x_1 = -h$. These results also show that the transient solution of crack opening displacement will jump and approach the corresponding static value after the Rayleigh wave has passed. In the case of $l = -h$, the phenomenon of crack surface interpenetration will not be eliminated during the transient history. But in the case of $l = h$, the crack surface will be opened again after the Rayleigh wave has passed.

It seems that the interpenetration is severe enough to invalidate the present transient results, but this phenomenon changes rapidly when the reflected longitudinal wave generated from the boundary of the half-plane reaches the crack surface. The completed structure of the crack opening displacement, induced by the influence of the reflected waves from the free boundary, is represented in Figs 11 and 12. Each reflected wave (PP, PS) induced at the half-plane boundary will generate reflected waves (R), diffracted waves (DP, DS) and Rayleigh waves (SR) at the crack surface. It is shown in Figs 11 and 12 that the crack surface will be opened again after the reflected PP wave returns to the crack surface and the time taken for the reflected wave to arrive at the crack in different cases is denoted by the arrow line. Even though the crack surface interpenetration is severe the duration time is small enough to neglect this effect in most cases.

6. CONCLUSION

In this study, an exponentially distributed loading on the crack surfaces in the Laplace transform domain is considered as the fundamental solution. The waves resulting from the diffraction of incident waves from the explosive source by the crack tip and reflected waves from the free surface can be constructed by superimposing the fundamental solution. This new methodology is shown to be both powerful and efficient in solving more complex and difficult problems.

In the previous sections, a subsurface inclined crack subjected to a longitudinal wave generated by an explosive source has been investigated. The net result of this loading will induce a mixed mode field at the crack tip. Exact mixed mode I and II stress intensity factors and crack opening displacement are obtained in an explicit form. The present results have been justified by the corresponding static value without considering the effect of the half-plane boundary. The crack surface interpenetration may occur, which will depend on the location of the dilatational source and crack orientation. In general, the crack surface will be opened after the reflected waves generated from the free half-plane boundary arrive at the crack surface. The exact solution to this configuration can provide a valuable check for pure numerical methods such as finite element, finite difference or boundary element methods in solving more complex geometries. The maximum circumferential tensile stress and minimum strain energy criteria are used to predict the direction of crack propagation. It is found in this study that the crack will most likely extrude out of the half-plane surface.

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